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On the Oberwolfach problem for complete multigraphs

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Abstract

In this paper we solve a uniform length cycle version of the Oberwolfach problem for multigraphs by giving necessary and sufficient conditions for the existence of a 2-factorization of λK_{dm} or $\lambda K_{dm} - I$ into 2-factors consisting of m cycles only.

1. Introduction

The Oberwolfach problem is due to Ringel [4], and, as originally posed, asks whether or not a complete graph on an odd number of vertices can be partitioned into subgraphs, each isomorphic to a given 2-factor. Huang et al. [7] considered the analogous question for complete graphs on an even numbers of vertices with a 1-factor removed (the vertices of the graph have to be of even degree for the question to make sense). The problem has been solved for the case when the given 2-factor consists of cycles of equal length [1,2,6]. In the present paper we extend these results to multigraphs, namely, we give sufficient and necessary conditions on λ, m, and d so that λK_{dm} (or λK_{dm} minus a 1-factor if dm is even) has a 2-factorization, with each 2-factor containing only m-cycles.

We start with some definitions. In this paper we deal with *multigraphs*, that is, multiple edges are allowed but loops are not. For a multigraph G, we denote by $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. If $H_1, H_2, \ldots, H_n$ are edge disjoint submultigraphs of G whose edge-sets form a partition of $E(G)$, then we write

$$G = H_1 \oplus H_2 \oplus \cdots H_n. \tag{1}$$

In the particular case of H_i being all isomorphic to the same graph H (with isomorphisms between H_i induced by the identity mapping on $V(G)$), we use the notation $G = nH$ instead of (1).

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We define the *wreath product* of multigraphs G and H to be the multigraph $G \wr H$ obtained from G by replacing each vertex of G by a copy of H, and, for every edge e of G, joining the two sets of vertices of $G \wr H$ corresponding to the endpoints of e by a complete bipartite graph. By K_n and $\overline{K}_n$ we denote the complete graph on n vertices and the edgeless graph on n vertices, respectively.

A *spanning submultigraph* F of a multigraph G is one which satisfies $V(F)=V(G)$. If every vertex has degree n in F, then F is an *n-factor* of G. In such a case we will sometimes also refer to the set of the edges of F as the n-factor. In particular, if F is a 1-factor, then $G-F$ will denote the multigraph obtained from G by the removal of the edges of F. A *2-factorization* of G is a partition of $E(G)$ into 2-factors. A *resolvable* $\{m\}$*-cycle decomposition* is a 2-factorization of G into 2-factors, each of which contains only m-cycles. To shorten the notation, we will denote such a 2-factorization by $\{m\}$-RCD.

2. The result

The proof of our result will be split into proofs of a number of lemmas, many of them quite technical. Among these, Lemma 1 is perhaps the most interesting one.

Lemma 1. *Let $m \geqslant 5$ be an odd integer. Then $2K_{2m}$ has a 2-factorization with each factor comprising two m-cycles.*

Proof. $2K_{2m}$ can be written as $G_1 \oplus G_2$, where G_1 and G_2 are copies of the complete graph on $2m$ vertices.

If I and J are 1-factors of G_1 and G_2, respectively, then G_1-I and G_2-J have $\{m\}$-RCD's $\mathscr{F}_1$ and $\mathscr{F}_2$ by Lemma 3.4 and Theorem 4.1 of [7]. Also, the constructions in [7] are such that $\mathscr{F}_1$ contains a 2-factor $F=C_1 \cup C_2$ (C_1 and C_2 are m-cycles) such that at least one of the edges of I has both of its endpoints in C_1. We will show that J can be chosen such that $C_1 \cup C_2 \cup I \cup J$ has an $\{m\}$-RCD $\mathscr{F}_3$. $\mathscr{F}_3 \cup (\mathscr{F}_1 \setminus \{F\}) \cup \mathscr{F}_2$ is then an $\{m\}$-RCD of $2K_{2m}$.

Let

$$C_1 = u_1u_2\ldots u_mu_1,$$

$$C_2 = v_1v_2\ldots v_mv_1.$$

Also, let $I=\{e_1,e_2,\ldots,e_m\}$. Since m is odd, I must also contain an edge with one endpoint in C_1, and the other in C_2. Thus, we may assume that $e_1=u_1v_1$, and $e_2=u_2u_s$, for some $4 \leqslant s \leqslant m$.

Let $f_1=u_1u_{s-1}$ be an edge in G_2. Then

$$C' = (C_1 \cup \{e_2, f_1\}) \setminus \{u_1u_2, u_{s-1}u_s\}$$

is an m-cycle (see Fig. 1; C_1' is drawn thick), and $C_1' \cup C_2$ is a 2-factor in $2K_{2m}$, consisting of two m-cycles.

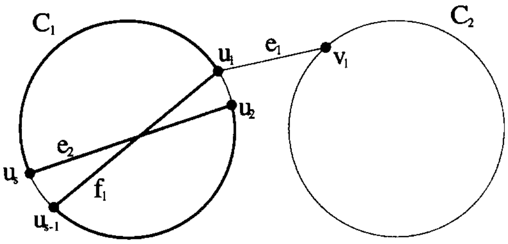

Fig. 1.

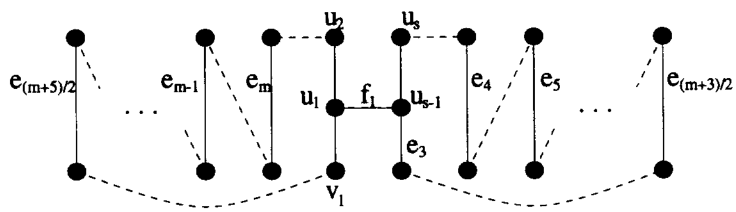

Fig. 2.

It remains to show that $\{f_1\}$ can be extended to a 1-factor J of G_2 so that

$$(J \cup I \cup \{u_1u_2, u_{s-1}u_s\}) \setminus \{e_2, f_1\} \tag{2}$$

is also a 2-factor comprising two m-cycles. Now, $(I \cup \{u_1u_2, u_{s-1}u_s\}) \setminus \{e_2\}$ is a collection of $m-1$ pairwise disjoint paths, two of them having length 2, and all the remaining ones being of length 1 (see solid lines in Fig. 2). Also, f_1 joins the midpoints of the two paths of length 2. It is easy to see that $\{f_1\}$ extends to a 1-factor J such that (2) holds (see Fig. 2 — the edges of J are the dashed lines). $\square$

Lemma 2. *If $p \geqslant 5$ is prime, $d \neq 4$ is even and $(p, d) \neq (5, 6)$, then $2K_{dp}$ has a $\{p\}$-RCD.*

Proof. For p and d as in the assumption, we have $2K_{dp} = \frac{1}{2}d(2K_{2p}) \oplus 2((K_d - I) \wr \overline{K}_p)$.

Lemma 1 implies that $\frac{1}{2}d(2K_{2p})$ has a $\{p\}$-RCD. The Lemmas 39 and 40 from [2] imply that $2((K_d - I) \wr \overline{K}_p)$ has a $\{p\}$-RCD. This completes the proof. $\square$

Lemma 3. *The multigraph $2K_{30}$ has a $\{5\}$-RCD.*

Proof. Let the set of vertices of $2K_{30}$ be $Z_{29} \cup \{\infty\}$. Let

$$\begin{aligned} R = \{&(0,9,15,28,14),(27,10,20,11,16),\\ &(3,5,1,7,6),(26,8,18,25,13),\\ &(2,4,12,19,23),(\infty,17,22,21,24)\}. \end{aligned}$$

It is easily checked that R is a 2-factor and that all differences $a-b$ (where $a-\infty, \infty-a$ are considered to be $-\infty$ and ∞, respectively) with a,b being 2 consecutive vertices on one of the cycles of R (ordered so that $a-b \in \{1,\dots,14,\infty\}$) give every value from $\{1,\dots,14,\infty\}$ exactly twice. This implies that $R, R+1, R+2, \dots, R+28$ (where $R+i$ is obtained from i by substituting $a+i$ for a, $a \in Z_{29}$) is a $\{5\}$-RCD of $2K_{30}$. □

Lemma 4. *Let d and m, $m \geqslant 3$, be such that $2K_{dm}$ has an $\{m\}$-RCD. Then $2K_{dmn}$ has an $\{mn\}$-RCD for every odd integer n.*

Proof. Let $\{F_1, F_2, \dots, F_{dm-1}\}$ be an $\{m\}$-RCD of $2K_{dm}$. Now

$$2K_{dmn} = 2K_{dm} \wr 2K_n = F_1 \wr K_n \oplus F_2 \wr K_n \oplus F_3 \wr \overline{K}_n \oplus F_4 \wr \overline{K}_n \oplus \cdots \oplus F_{dm-1} \wr \overline{K}_n.$$

Each component of $F_1 \wr K_n, F_2 \wr K_n$, and $F_i \wr \overline{K}_n$, $i > 2$, is isomorphic to $C_m \wr K_n, C_m \wr K_n$, and $C_m \wr \overline{K}_n$, respectively. As n is odd, all these components decompose into Hamilton cycles by [3]. This implies that $2K_{dmn}$ has an $\{mn\}$-RCD. □

Lemma 5. *The multigraph $2K_{4m}$ has an $\{m\}$-RCD for $m = 5, 7, 9, 11$ and 13.*

Proof. Given m, let the vertices be the elements of $Z_{4m-1} \cup \{\infty\}$. Similarly as in the proof of Lemma 3, $R, R+1, R+2, \dots, R+(4m-2)$ form an $\{m\}$-RCD of $2K_{4m}$. The respective R's are:

$$\begin{aligned}
m=5:\quad R &= \{(0,9,1,8,2),(18,11,17,12,16),(7,3,6,5,10),\\
&\quad (15,14,\infty,13,4)\}\\
m=7:\quad R &= \{(0,13,1,12,2,11,3),(26,15,25,16,24,17,22),\\
&\quad (10,4,9,5,7,14,8),(23,20,21,6,19,18,\infty)\}\\
m=9:\quad R &= \{(0,17,1,16,2,15,3,14,4),\\
&\quad (34,19,33,20,32,21,31,22,29),\\
&\quad (13,5,12,6,11,7,9,18,10),\\
&\quad (30,24,8,25,23,26,27,28,\infty)\}\\
m=11:\quad R &= \{(0,21,1,20,2,19,3,18,4,17,5),\\
&\quad (42,23,41,24,40,25,39,26,38,27,36),\\
&\quad (16,6,15,7,14,8,13,9,11,22,12),\\
&\quad (37,34,33,\infty,35,28,30,10,31,32,29)\}\\
m=13:\quad R &= \{(0,25,1,24,2,23,3,22,4,21,5,20,6),\\
&\quad (50,27,49,28,48,29,47,30,46,31,45,32,43),\\
&\quad (19,7,18,8,17,9,16,10,15,11,13,26,14),\\
&\quad (44,40,39,\infty,42,33,41,38,12,36,35,37,34)\}.\quad \square
\end{aligned}$$

Lemma 6. *The multigraph $2K_{4m}$ has an $\{m\}$-RCD for all $m \geqslant 15$, m odd.*

Proof. We will again use the same idea as in the previous lemmas. That is, the vertices will be the elements of $Z_{4m-1} \cup \{\infty\}$ and R will consist of four m-cycles which will be chosen so that among all differences $a-b$ every member of $\{1,2,\dots,2m-2,\infty\}$ will be present exactly twice. Let k be such that $m = 2k+1$ and let the first three cycles of R be as follows:

$$(0, 4k+1, 1, 4k, 2, 4k-1, \dots, k-1, 3k+2, k),$$

$$(8k+2, 4k+3, 8k+1, 4k+4, \dots, 7k+3, 5k+2, 7k+1),$$

$$(3k+1, k+1, 3k, k+2, 3k-1, k+3, \dots, 2k+3, 2k-1, 2k+1, 4k+2, 2k+2).$$

These three cycles yield, respectively, the following three sequences of differences:

$$k, 2k+2, 2k+3, 2k+4, \dots, 4k+1,$$

$$k+1, 2k-1, 2k+1, 2k+2, \dots, 4k-1,$$

$$2, 4, 5, 6, 7, 8, 9, \dots, 2k-1, 2k, 2k+1, k-1, 2k.$$

The fourth cycle is described in what follows. First, use $k-5$ edges to construct three paths (their respective lengths are $\lceil \frac{1}{3}(k-5) \rceil$, $\lceil \frac{1}{3}(k-6) \rceil$, and $\lfloor \frac{1}{3}(k-5) \rfloor$):

$$5k+3, 7k-1, 5k+6, 7k-4, \dots, a_1,$$

$$7k, 5k+5, 7k-3, 5k+8, \dots, a_2,$$

$$5k+4, 7k-2, 5k+7, 7k-5, \dots, a_3$$

(the differences produced are $k+2, k+3, \dots, 2k-4$). Let i, j and l be so that $a_i < a_j < a_l$. There are two possibilities for the values of a_1, a_2 and a_3:

1. $a_i + 1 = a_j$
2. $a_j + 1 = a_l$.

1. Construct these three paths:

$$a_i + 4, a_i + (k+1), a_i + 7, a_i + (k-2), \dots, b_i,$$

$$a_j + 4, a_j + (k-1), a_j + 7, a_j + (k-4), \dots b_j,$$

$$a_l, a_l - (k-4), a_l - 3, a_l - (k-7), \dots, b_l.$$

2. In this case the paths are:

$$a_i + 4, a_i + k, a_i + 7, a_i + (k-3), \dots b_i,$$

$$a_j, a_j - (k-5), a_j - 3, a_j - (k-8), \dots, b_j,$$

$$a_l, a_l - (k-3), a_l - 3, a_l - (k-6), \dots, b_l.$$

In both cases $k-7$ edges are used (here we need $k \geqslant 7$ but this is assured by $m \geqslant 15$) yielding the differences $5, 6, 7, \ldots, k-3$. It is easily checked that in both cases $\{b_i, b_j, b_l\} = \{6k+1, 6k+2, 6k+6\}$. Here we distinguish three cases:

(a) $b_l = 6k+1$,

(b) $b_l = 6k+2$,

(c) $b_l = 6k+6$.

In the cases 1(a)–1(c) add these edges: $(a_i, a_i+4), (a_j, a_j+2), (a_j+1, a_j+2), (a_j+1, a_j+4), (6k+3, 6k+6), (6k+4, 6k+5)$ and, moreover,

$$(2k, 6k+1), (2k, 6k+3), (6k+2, \infty), (6k+5, \infty) \quad \text{in } 1(\text{a}),$$

$$(2k, 6k+2), (2k, 6k+3), (6k+1, \infty), (6k+5, \infty) \quad \text{in } 1(\text{b}),$$

$$(2k, 6k+1), (2k, 6k+3), (6k+2, \infty), (6k+5, \infty) \quad \text{in } 1(\text{c}).$$

In the cases 2(a)–(c) add the edges $(a_i, a_i+3), (a_i+2, a_i+3), (a_i+1, a_i+2), (a_i+1, a_i+4)$ and, moreover,

$$\begin{array}{llll} (6k+4, 6k+6) & (6k+1, 6k+5) & (2k, 6k+2) & \\ (2k, 6k+3) & (6k+3, \infty) & (6k+5, \infty) & \text{in } 2(\text{a}), \\ (6k+3, 6k+5) & (6k+2, 6k+6) & (2k, 6k+1) & \\ (2k, 6k+3) & (6k+4, \infty) & (6k+5, \infty) & \text{in } 2(\text{b}), \\ (6k+3, 6k+5) & (6k+2, 6k+6) & (2k, 6k+1) & \\ (2k, 6k+3) & (6k+4, \infty) & (6k+5, \infty) & \text{in } 2(\text{c}). \end{array}$$

In each of these six cases add at the end the edges $(5k+3, 7k), (5k+4, 7k+2), (6k+4, 7k+2)$.

All these edges yield, in every case, the differences $1, 2, 3, 4, k-2, 2k-3, 2k-2, 4k, 4k+1, \infty, 1, 3, \infty$.

Now the union of all used paths and edges form the fourth m-cycle of R. It is tedious but straightforward to check that R is indeed a 2-factor satisfying the required conditions. The $\{m\}$-RCD of $2K_{4m}$ is formed by $R, R+1, R+2, \ldots R+(4m-2)$. $\square$

Lemma 7. *Let d be even and $m \geqslant 5$ be odd. Then $2K_{dm}$ has an $\{m\}$-RCD.*

Proof. If $d=4$, then the result follows by Lemmas 5 and 6. So we may assume that $d \neq 4$. If m is a prime, then Lemmas 2 and 3 yield an $\{m\}$-RCD of $2K_{dm}$. Otherwise distinguish two cases:

(1) m is not a power of 3,

(2) m is a power of 3.

(1) $m = pm'$ where $p \geqslant 5$ is a prime. By one of Lemmas 2 and 3 there is a $\{p\}$-RCD of $2K_{dp}$. By Lemma 4, $2K_{dm}$ has an $\{m\}$-RCD .

(2) If $d \neq 2$, then $2K_{3d}$ has a $\{3\}$-RCD by [5]. Lemma 4 implies again that $2K_{dm}$ has an $\{m\}$-RCD. Hence we may assume $d = 2$, and the result follows from Lemma 1. □

Before we embark on settling the question whether λK_{dm} (or $\lambda K_{dm} - I$) has an $\{m\}$-RCD we need two more lemmas:

Lemma 8. *The multigraph $\lambda K_6 - I$ has no* $\{3\}$-RCD whenever λ is odd.

Proof. Let the vertices of $\lambda K_6 - I$ be denoted by $0_0, 1_0, 2_0, 0_1, 1_1$ and 2_1. Without loss of generality we may assume that $I = \{(0_0, 0_1), (1_0, 1_1), (2_0, 2_1)\}$. Then $\lambda K_6 - I$ contains 3λ edges of one of the types $(0_0, 1_1), (1_0, 2_1)$ and $(2_0, 0_1)$. It is easy to observe that any 2-factor of $\lambda K_6 - I$ consisting of two 3-cycles contains an even number of edges that are of one of the above types. Hence for $\{3\}$-RCD of $\lambda K_6 - I$ to exist 3λ would have to be even. As this is not the case the lemma is proven. □

Lemma 9. *The multigraph $3K_{12} - I$ has a* $\{3\}$-RCD.

Proof. Let the vertices of $3K_{12} - I$ be denoted by i_j, $0 \leqslant i \leqslant 5$, $0 \leqslant j \leqslant 1$ and let $I = \{(i_0, i_1) : 0 \leqslant i \leqslant 5\}$. For a subgraph H of $3K_{12} - I$ denote by $H + s$ the subgraph of $3K_{12} - I$ given by $H + s = \{((i+s)_j, (k+s)_l) : (i_j, k_l) \in H\}$. Let

$$R_1 = \{(0_1, 1_1, 2_1), (0_0, 5_1, 5_0), (1_0, 3_1, 4_0), (2_0, 3_0, 4_1)\},$$

$$R_2 = \{(0_1, 1_1, 4_0), (2_1, 5_1, 3_0), (3_1, 4_1, 5_0), (0_0, 1_0, 2_0)\},$$

$$R_3 = \{(0_1, 1_1, 0_0), (2_1, 5_1, 4_0), (3_1, 4_1, 3_0), (1_0, 2_0, 5_0)\},$$

$$R_4 = \{(0_1, 3_1, 0_0), (1_1, 5_1, 2_0), (2_1, 5_0, 4_0), (4_1, 1_0, 3_0)\},$$

$$R_5 = \{(0_1, 2_1, 3_0), (1_1, 3_1, 5_0), (4_1, 0_0, 2_0), (5_1, 1_0, 4_0)\},$$

$$R_6 = \{(0_1, 2_1, 4_1), (1_1, 3_1, 5_1), (0_0, 2_0, 4_0), (1_0, 3_0, 5_0)\}.$$

Then $R_i, R_i + 2, R_i + 4, 1 \leqslant i \leqslant 5$ together with R_6 form the desired $\{3\}$-RCD. □

At this point, we are ready to prove our main theorem. We first list the results that deal with graphs (i.e., $\lambda = 1$) that will be used in our proof.

Theorem 10 (Alspach and Häggkvist [1]). *If $m \geqslant 4$ is even, then $K_{dm} - I$ has an* $\{m\}$-RCD *for all positive d.*

Theorem 11 (Alspach et al. [2]). *If $m \geqslant 3$ is odd, then K_{dm} (or $K_{dm} - I$ when d is even) has an* $\{m\}$-RCD *for all positive $d, d \neq 4$, except for the case $d = 2, m = 3$.*

Theorem 12 (Hoffman and Schellenberg [6]). *If $m \geqslant 5$ is odd, then $K_{4m} - I$ has an* $\{m\}$-RCD.

The main theorem now follows:

Theorem 13. *Let λ, d and m be positive integers with $m \geqslant 3$. Then λK_{dm} (or $\lambda K_{dm} - I$ when λ is odd and dm is even) has an $\{m\}$-RCD if and only if none of the following is the case:*

(i) $\lambda \equiv 2 \pmod 4$, $d = 2$, $m = 3$
(ii) λ *odd*, $d = 2$, $m = 3$
(iii) $\lambda = 1$, $d = 4$, $m = 3$

Proof. The nonexistence of an $\{m\}$-RCD follows in case (ii) from Lemma 8 and in case (iii) from [8]. In case (i), if an $\{m\}$-RCD existed, it would contain an odd number of 2-factors. If u, v, w is any triangle of λK_6, each 2-factor would use either one edge or three edges of u, v, w. But this is impossible since λ is even.

Therefore, it suffices to show that in all other possible cases there is an $\{m\}$-RCD. We will distinguish eight distinct cases altogether depending on whether λ, d and m are even or odd.

Case 1: λ *even*, d *even*, m *even*. Now

$$\lambda K_{dm} = \tfrac{1}{2}\lambda((K_{dm} - I_1) \oplus (K_{dm} - I_2) \oplus I_1 \oplus I_2),$$

where I_1, I_2 are arbitrary 1-factors in K_{dm}. By Theorem 10, $K_{dm} - I_1$ and $K_{dm} - I_2$ both have an $\{m\}$-RCD. As m is even and I_1, I_2 are arbitrary they can be chosen so that $I_1 \oplus I_2$ consists of d m-cycles.

Case 2: λ *even*, d *even*, m *odd*. If $m \geqslant 5$, then $2K_{dm}$ has an $\{m\}$-RCD by Lemma 7. As $\lambda K_{dm} = \frac{1}{2}\lambda(2K_{dm})$, λK_{dm} has an $\{m\}$-RCD. If $m = 3$, then by [5] λK_{dm} has an $\{m\}$-RCD unless $\lambda \equiv 2 \pmod 4$ and $d = 2$.

Case 3: λ *even*, d *odd*, m *even*. This case is the same as Case 1.

Case 4: λ *even*, d *odd*, m *odd*. Now K_{dm} (and, therefore, λK_{dm}) has an $\{m\}$-RCD by Theorem 11.

Case 5: λ *odd*, d *even*, m *even*. Now

$$\lambda K_{dm} - I = (\lambda - 1)K_{dm} \oplus (K_{dm} - I).$$

The case for $(\lambda - 1)K_{dm}$ is similar to Case 1 and $K_{dm} - I$ has an $\{m\}$-RCD by Theorem 10.

Case 6: λ *odd*, d *even*, m *odd*. Here

$$\lambda K_{dm} - I = (\lambda - 1)K_{dm} \oplus (K_{dm} - I).$$

The case for $(\lambda - 1)K_{dm}$ is essentially the same as Case 2 and, hence, it has an $\{m\}$-RCD unless $\lambda \equiv 3 \pmod 4), d = 2, m = 3$. $K_{dm} - I$ has, by Theorems 11 and 12, an $\{m\}$-RCD unless $d = 4, m = 3$ or $d = 2, m = 3$. Combining these conditions we see that $\lambda K_{dm} - I$ has an $\{m\}$-RCD except possibly when $m = 3, d \in \{2, 4\}$. We already saw that there is no $\{m\}$-RCD when $d = 2, m = 3$ and when $d = 4, m = 3, \lambda = 1$. If $d = 4, m = 3$ and $\lambda \geqslant 3$, we have $\lambda K_{dm} - I = (\lambda - 3)K_{4\cdot 3} \oplus (3K_{12} - I)$, and $(\lambda - 3)K_{4\cdot 3}$ was shown to have a $\{3\}$-RCD (Case 2) and $3K_{12} - I$ has a $\{3\}$-RCD by Lemma 9.

Case 7: λ *odd*, d *odd*, m *even*. In this case

$$\lambda K_{dm} - I = (\lambda - 1)K_{dm} \oplus (K_{dm} - I).$$

The multigraph $(\lambda - 1)K_{dm}$ is covered by Case 3 and $K_{dm} - I$ has an $\{m\}$-RCD by Theorem 10.

Case 8: λ *odd*, d *odd*, m *odd*. By Theorem 11, K_{dm} has an $\{m\}$-RCD.

This exhausts all possible cases and Theorem 13 is proved. □

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